IV Existance of Contact Structures

let M be a 3-manifold

K a knot in M

$N = S^1 \times D^2$ a neighborhood of K in M

$\alpha \subset \partial N = \partial \overline{M \setminus N}$ an embedded curve

$f: \partial(S^1 \times D^2) \to \partial \overline{M \setminus N}$ a diffeomorphism sending $\{pt\} \times \partial D^2$ to α

α-Dehn surgery on M along K is

$$M(K, \alpha) = (\overline{M \setminus N}) \cup_f (S^1 \times D^2)$$

glue $p \in \partial(S^1 \times D^2)$ to $f(p) \in \partial(\overline{M \setminus N})$

Facts: 1) if f_1, f_2 are 2 diffeomorphisms that send $\{pt\} \times \partial D^2$ to α

then $(\overline{M \setminus N}) \cup_{f_1} (S^1 \times D^2) \cong (\overline{M \setminus N}) \cup_{f_2} (S^1 \times D^2)$

i.e. $M(K, \alpha)$ well-defined

2) any closed 3-manifold M is obtained from S^3 by by Dehn surgery on some link

exercise: prove (or look up) these facts

note: $T^2 = \partial N \subset \partial(\overline{M \setminus N})$

there is a curve $\mu \subset \partial(\overline{M \setminus N})$ that bounds a disk in $N = S^1 \times D^2$

if K is null-homologous then there is a curve $\lambda \in \partial(\overline{M \setminus N})$ such that $\lambda = \partial \Sigma$, Σ a surface in $\overline{M \setminus N}$

exercise: $|\lambda \cap \mu| = 1$

otherwise choose any λ on $\partial(\overline{M \setminus N})$ st. $|\lambda \cap \mu| = 1$

note: λ determines a framing of K

exercise: $[\lambda], [\mu]$ form a basis for $H_1(S^1 \times S^1) = \mathbb{Z} \oplus \mathbb{Z}$
$ \overset{\shortparallel}{\partial(\overline{M \setminus N})}$

so any $h \in H_1(S^1 \times S^1)$ can be written $q[\lambda] + p[\mu]$

exercise: h can be realized by an embedded curve iff p, q are relatively prime

now any diffeomorphism $\phi : \partial(S^1 \times D^2) \to \partial(\overline{M \setminus N})$ is determined upto isotopy by its action on
$H_1(\partial(S^1 \times D^2)) \to H_1(\partial(S^1 \times D^2))$
i.e. by a matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ with $ad - bc = 1$

$\mathbb{Z}$-valued

exercise: Prove this

Hint: a simple closed curve on T^2 is determined, upto isotopy, by its class in $H_1(T^2)$
if 2 diffeomorphisms do the same thing on $(S^1 \times \{pt\}) \cup (\{pt\} \times S^1)$ then they are isotopic (use any diffeomorphism of ∂D^2 extends to D^2)

if α is an embedded curve in $\partial(\overline{M \setminus N})$ then
$[\alpha] = q[\lambda] + p[\mu]$

so diffeomorphism sending $\{pt\} \times \partial D^2$ to α is
$\begin{bmatrix} r & q \\ s & p \end{bmatrix}$ with $rp - qs = -1$

$\swarrow = \infty$ if $q = 0$

we denote $M(K,\alpha)$ by $M_K(p/q)$

Th$^{\underline{m}}$ 1 (Martinet):

$\boxed{\text{Any closed oriented 3-manifold admits a contact structure}}$

Proof: given closed oriented 3-manifold M

from above we know M is obtained from S^3 by Dehn surgery on some link in S^3

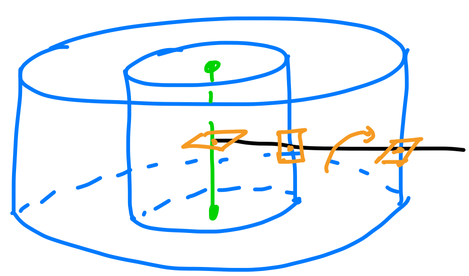
we address case where $M = S^3_K(p/q)$ but the general case will clearly follow

$1^{\underline{st}}$ can isotop K so that it is transverse to ξ_{std}, by lemma II.3

$2^{\underline{nd}}$ K has a standard neighborhood by Th$^{\underline{m}}$ II.3

that is let $U = S^1 \times \mathbb{R}^2 = \mathbb{R}^3 / z \mapsto z+1$

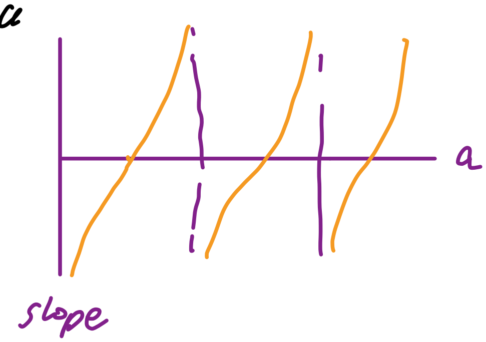
with $\xi = \ker(\cos r \, dz + r \sin r \, d\theta)$



let $S_a = \{(r,\theta,z) \mid r \leq a\}$, $T_a = \partial S_a$

K has a neighborhood N s.t. N contactomorphic to S_a for some a (any a close to 0)

note: $(T_a)_\xi$ is nonsingular and has slope $-\frac{1}{a}\cot a$



so $(\partial \overline{S^3\setminus N})_{\xi_{std}}$ has slope a' some a'

3^{rd} glue in standard contact torus

specifically $\overline{S^3\setminus N}$ has contact structure $\xi_{std}|_{\overline{M\setminus N}}$

we glue $S^1\times D^2$ to $\overline{S^3\setminus N}$ via

$$f: \partial(S^1\times D^2) \to \partial(\overline{S^3\setminus N})$$

f^{-1} take fol^{n} of slope a' on $\partial \overline{S^3\setminus N}$ to a fol^{n} of slope b' on $\partial(S^1\times D^2)$

now let b be such that $-\frac{1}{b}\cot b = b'$

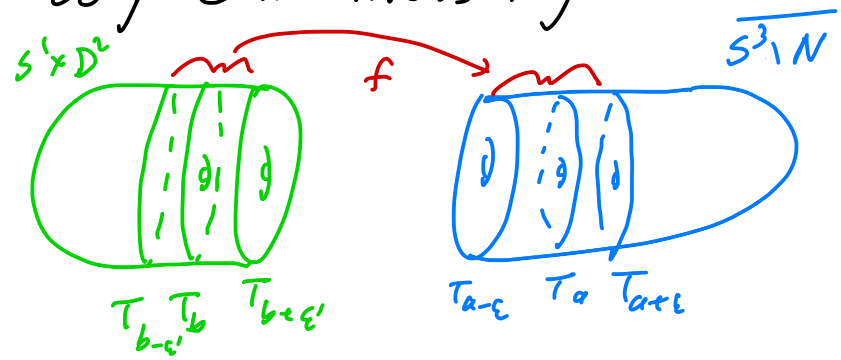
identify $S^1\times D^2$ with S_b

note: f takes $(\partial S_b)_\xi$ to $(\partial(\overline{S^3\setminus N}))_{\xi_{std}}$

so Th^{m} II.5 says f can be isotoped to be a contactomorphism

so can glue $(\overline{S^3\setminus N}, \xi_{std})$ and (S_b, ξ) via f to get a contact structure on M!

to be more rigorous should take $\overline{S^3 \setminus S_{a-\varepsilon}}$ and $S_{b+\varepsilon'}$ so there is a colar neighborhood to glue manifolds together



extend f to $T^2 \times I$ and make contactomorphism here

exercise: Give a second proof of existence of contact structures using the fact that all close oriented 3-manifolds are covers of S^3 branched over some link

Hint: make branch locus transverse.

we can use this construction to do better!

let $\text{Dist}(M) = \{\text{all oriented plane fields on } M\}$

$\cong \{\text{all oriented line fields on } M\}$

fix a metric

$\text{Cont}(M) = \{\text{all contact structures on } M\}$

C^∞-topology

$\text{Dist}(M)$ sections of $Gr_2(TM)$

Grassmann of 2-planes in TM

we have a natural inclusion map

$$\text{Cont}(M) \xrightarrow{i} \text{Dist}(M)$$

Th$^{\underline{m}}$ 2 (Lutz):

$$\boxed{i_*: \pi_0(\text{Cont}(M)) \longrightarrow \pi_0(\text{Dist}(M)) \text{ is } \underline{\text{onto}}}$$

this says every plane field is homotopic to a contact structure

<u>Major Question:</u> Is i_* injective?

If not understand $i_*^{-1}(x)$ for $x \in \pi_0(\mathrm{Dist}(M))$

Before proving this theorem we need to better understand $\mathrm{Dist}(M)$, how big is $\pi_0(\mathrm{Dist}(M))$?

<u>Fact:</u> if M is a closed oriented 3-manifold then

$$T^*M \cong M \times \mathbb{R}^3$$

you can see a proof of this in Kirby's book "The topology of 4-manifolds"

fix a metric g (this is not really necissary)

$$U(TM) = \text{unit tangent bundle} = M \times S^2$$

$$\mathrm{Dist}(M) = \{\text{oriented plane field}\} \xrightarrow[\cong]{g} \{\text{unit vector field}\} = \Gamma(U(TM))$$

$\xi \longmapsto v$ where $v(x)$ = unit positive orthogonal to ξ_x

given $v \in \Gamma(U(TM))$ we have

$$v: M \to M \times S^2$$
$$p \mapsto (p, f_v(p))$$

so v determined by $f: M \to S^2$

So we have $\text{Dist}(M) \underset{\text{1-1 corresp.}}{\longleftrightarrow} \{\text{maps } M \to S^2\}$

(correspondence depends on trivialization of TM, but not metric)

so $\pi_0(\text{Dist}(M))$ = homotopy classes of maps $M \to S^2$
$=: [M, S^2]$

example: $[S^3, S^2] \cong \pi_3(S^2) \cong \mathbb{Z}$
generated by the Hopf map

remark: we will see below that $\pi_0(\text{Dist}(M))$ is always infinite so Thm 2 say all orientable 3-manifold admit infinitely many different contact structures!

a framed submanifold $(N, \mathcal{F})$ of a manifold X is a submanifold $N \subset X$ together with a trivialization $\mathcal{F}$ of the normal bundle of N in X

$(N_i, \mathcal{F}_i)$, $i=0,1$, in X are framed cobordant if there is a framed submanifold $(N', \mathcal{F}')$ of $X \times [0,1]$ such that

$$(N', \mathcal{F}') \cap (X \times \{i\}) = (N_i, \mathcal{F}_i)$$

lemma 3 (Thom-Pontryagin construction in 3-D):

$$\boxed{[M^3, S^2] \underset{\text{1-1 correspondence}}{\longleftrightarrow} \{\text{framed cobordism classes of 1-mfds in } M\} =: \Omega_1^f(M)}$$

Proof: given $\phi : M \to S^2$

can homotop ϕ so ϕ is transverse to north pole $n \in S^2$

let $\gamma = \phi^{-1}(n)$ this is a 1-mfd in M

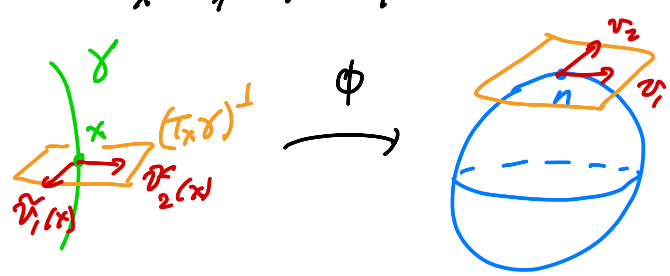
moreover note $d\phi_x : T_x M \to T_n S^2$ is onto since

ϕ transverse to n

fix a basis v_1, v_2 for $T_n S^2 = \mathbb{R}^2$

now $\tilde{v}_1(x), \tilde{v}_2(x)$ perpendicular to $T_x\gamma$ (fix metric)

s.t. $d\phi_x(\tilde{v}_1(x)) = v_1$



(recall $T_x\gamma = \ker(d\phi_x)$ and $d\phi_x|_{(T_x\gamma)^\perp}$ is isomorphism)

so $\tilde{v}_1, \tilde{v}_2$ are two linear independent sections of $\nu(\gamma)$ they give a framing $\mathcal{F}$ to γ

so $[M, S^2] \xrightarrow{\Psi} \Omega_1^f(M)$ is a map

$\phi \longmapsto (\gamma, \mathcal{F})$

exercise: show Ψ is well-defined

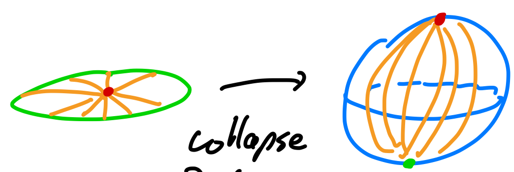
now given $(\gamma, \mathcal{F}) \in \Omega_1^f(M)$ we need to find a map ϕ

such that $\Phi(\phi) = (\gamma, \mathcal{F})$

note: γ has a neighborhood $N = \gamma \times D^2$ given by framing

now define $\phi : N \to S^2$

to be



on each $\{pt\} \times D^2$ in $N = \gamma \times D^2$

define $\phi : \overline{M \setminus N} \to S^2 : p \mapsto$ south pole

ϕ can be constructed to be smooth on interior of N and perturbed to be smooth s.t. $\phi^{-1}(n)$ still γ

exercise: $\Phi(\phi) = (\gamma, \mathcal{F})$

so Φ surjective

now suppose $\Phi(\phi_0) = (\gamma_0, \mathcal{F}_0)$, $\Phi(\phi_1) = (\gamma_1, \mathcal{F}_1)$

$(\gamma_0, \mathcal{F}_1)$ framed cobordant via $(\Sigma, \mathcal{F})$

exercise: if $(\gamma_0, \mathcal{F}_0) = (\gamma_1, \mathcal{F}_1)$, then show ϕ_0 is homotopic to ϕ_1

exercise: in general, construct a homotopy $M \times [0,1] \to S^2$ from ϕ_0 to ϕ_1 using $(\Sigma, \mathcal{F})$ just as we constructed ϕ above so we know $\pi_0(\mathrm{Dist}) \overset{1\text{-}1}{\longleftrightarrow} \Omega_1^f(M^3)$

let's study $\Omega_1^f(M)$

set $\Omega_1(M) = \{$ cobordism classes of 1-manifolds in $M\}$

same as $\Omega_1^f(M)$ but forget framing

lemma 4:

$$\boxed{\Omega_1(M) \underset{1\text{-}1}{\longleftrightarrow} H_1(M)}$$

Proof: given $\gamma \in \Omega_1(M)$

we can "triangulate" (write as 1-complex)

so it gives a 1-cycle $\therefore$ an element of $H_1(M)$

if γ_0, γ_1 cobordant via surface $\Sigma \subset M \times [0,1]$

project surface to M, triangulate

to get a 2-chain in $C_2(M)$

as 1-chains

exercise: $\partial \Sigma = \gamma_1 - \gamma_2$

as 2-chain

so γ_1 homologous to γ_2

and $\Omega_1(M) \xrightarrow{\Phi} H_1(M)$ well-defined

any $h \in H_1(M)$ is represented by the image of an S^1,

so Φ clearly onto

now if $\Phi(\gamma_0) = \Phi(\gamma_1)$, then there is a 2-chain c st.

$$\partial c = \gamma_1 - \gamma_0$$

exercise: can find another 2-chain C' such that C' = image of triangulated surface Σ.

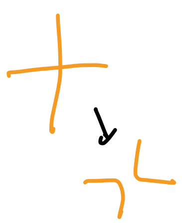
let $f: \Sigma \to [0,1]$ be smooth map st. $f^{-1}(i) = \gamma_i$.

now
$$\begin{aligned} \Sigma &\longrightarrow M \times [0,1] \\ p &\longmapsto (p, f(p)) \end{aligned}$$

is a map that can be perturbed to be smooth and self transverse

this means the image in $M \times [0,1]$ is an immersed surface with transverse double points

one may "resolve" the double points to get an embedded surface Σ' in $M \times [0,1]$ st. $\partial \Sigma' = \gamma_0 \cup \gamma_1$



$\therefore \gamma_1 = \gamma_0$ in $\Omega_1(M)$

exercise: fill in details of argument above

there is a natural map $F: \Omega_1^{f}(M) \to \Omega_1(M)$ that just forgets the framing

lemma 4:

given $x \in \Omega_1(M)$,

$$F^{-1}(x) = \mathbb{Z}/_{2d([\Sigma(x)])}$$ ← homology class of x

where $d(y)$ is the divisibility of y in $H_1(M)$ modulo torsion note: $0 \in H_1(M)$ for any M has divisibility 0

so $F^{-1}(0) = \mathbb{Z}$

$\therefore$ all M^3 have infinitely many homotopy classes of plane field and hence infinitely many contact structures by Th$^{\underline{m}}$ 2

Proof: given x a 1-submanifold in M

let $\mathcal{F}$ be a framing on x

and $\mathcal{F}_n$ is the framing on x given by adding n right handed twist to $\mathcal{F}$

the map $h: \mathbb{Z} \to F^{-1}(x)$ is clearly onto

suppose $h(n) = h(m)$

so there is a framed surface $(\Sigma, \mathcal{F}')$ in $M \times [0,1]$

s.t. $(\Sigma, \mathcal{F}') \cap (M \times \{0\}) = (x, \mathcal{F}_n)$

$(\Sigma, \mathcal{F}') \cap (M \times \{1\}) = (x, \mathcal{F}_m)$

let T = closed surface in $M \times S^1 = M \times [0,1] / _{M \times \{0\} \sim M \times \{1\}}$ given by Σ

exercise: Show $T \cdot T = m - n$

self-intersection

i.e have T transversely intersect a copy of T and count intersection points with sign

Hint: $\mathcal{F}'$ gives a way to push Σ off of itself to get Σ' think about how to make Σ' a closed surface in $M \times S^1$

let $C = x \times S^1 \subset M \times S^1$

note: $m - n = T \cdot T = [(T - C) + C] \cdot [(T - C) + C]$

$= (T-C)\cdot(T-C) + 2(T-C)\cdot C + \underbrace{C\cdot C}_{=0}$ use framing on x to get disjoint copy

Claim: $(T-C)\cdot(T-C) = 0$

indeed note $H_2(M\times S^1) = (H_2(M)\otimes H_0(S^1)) \oplus (H_1(M)\otimes H_1(S^1)) \oplus \cancel{H_0(M)\otimes H_2(S^1)} \to 0$

now $(T-C)\cap(M\times\{pt\}) = 0$

(since $T\cap(M\times\{pt\}) = x$
$C\cap(M\times\{pt\}) = x$)

so $T-C \in H_2(M)\otimes H_0(S^1)$

(since any non-zero elt in $H_1(M)\otimes H_1(S^1)$ has non-zero intersection with $M\times\{pt\}$)

so $T-C$ is homologous to $S\subset M$

$S\cdot S = 0$ (push copy of S in S^1 direction) ✓

so $m-n = (T-C)\cdot(2C)$
$= (T-C)\cdot(2x)$

since $T-C$ homologous to surface in M

so $m-n$ is divisible by $2d(x)$

Conversely, suppose $2d(x)\neq 0$ (ie x not torsion)

let y be a primitive class in $H_1(M)$ such that $x = d(x)y$

P.D.(y) is a generator of $H^2(M)$

Poincaré Dual

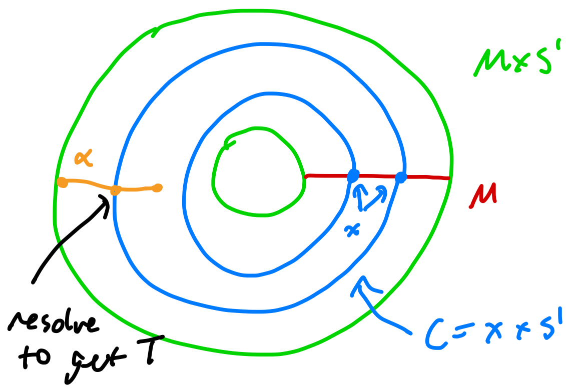
so $\exists$ a surface α such that

$$y \cdot \alpha = (P.D.(y))(\alpha) = 1$$

$\therefore \quad 2x \cdot \alpha = 2d(x)$

let T be a surface in $M \times S^1$ representing

$C + \alpha$



note: $(C+\alpha)\cdot(C+\alpha) = 2\, C\cdot\alpha = 2\, x\cdot\alpha = 2d(x)$

cut $M \times S^1$ along $M \times \{pt\}$ to get $M \times [0,1]$

and T becomes a cobordism from x to x and framings differ by $2d(x)$ (argue as above)

so $h: \mathbb{Z} \to F^{-1}(x)$ is onto with kernel $2d(x)\mathbb{Z}$

we now return to the proof of Th^m 2

Th^m 2 (Lutz):

$$\boxed{\iota_x : \pi_0(\mathrm{Cont}(M)) \longrightarrow \pi_0(\mathrm{Dist}(M)) \text{ is onto}}$$

we need a construction called Lutz twist

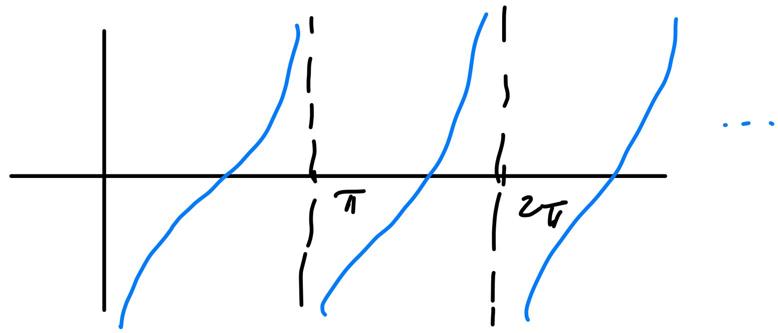
let $\mathbb{R}^2 \times S^1 = \mathbb{R}^3 / _{z \mapsto z + 2\pi}$

with contact structure $\xi = \ker(\cos r\, dz + r \sin r\, d\theta)$

$T = \{(0,0)\} \times S^1$ is a transverse curve

$S_a = \{ (r, \theta, z) \mid r \leq a \}$

$(\partial S_a)_\xi$ is linear foliation of slope $-\frac{1}{a}\cot a$



if K a transverse knot in (M, ξ), then it has a nbhd N contactomorphic to S_a for some $a \in (0, \pi)$

let $b \in (\pi, 2\pi)$ s.t. $(S_b)_\xi = (S_a)_\xi$

the result of replacing $N = S_a$ by S_b is called a half Lutz twist on T

replacing $N = S_a$ with S_c, for $c \in (2\pi, 3\pi)$ s.t. $(S_c)_\xi = (S_a)_\xi$, is called a full Lutz twist

note: Lutz twisting does not change M

lemma 5:

let (M, ξ') be the result of performing a half-Lutz twist on a transverse knot K in (M, ξ)

then $F(\xi') = F(\xi) - [K]$

where F is map from lemma 4 and $[K]$ is the homology class of K

if K is null-homologous and $(\gamma, \mathcal{F}) \in \Omega^f_1(M)$ corresponding to ξ then $(\gamma, \mathcal{F}_{sl(K)})$ corresponds to ξ' where $\mathcal{F}_n$ is framing $\mathcal{F}$ with n right-handed twists

Proof of Th$^{\text{m}}$2:

by Th$^{\text{m}}$1 $\exists$ some contact structure ξ on M

suppose ξ corresponds to $(\gamma, \mathcal{F})$

given some other $(\gamma', \mathcal{F}')$ let K be a transverse knot that realizes the homology class $[\gamma] - [\gamma']$

lemma 5 says the result of half Lutz twisting ξ along K is associated to the framed manifold $(\gamma', \mathcal{F}'')$

let U transverse unknot with $sl = -1$ (push off of )

T be tranverse right handed trefoil with $sl = 1$ (push off of )

half Lutz twisting on U does not change γ' but sends $\mathcal{F}''$ to $\mathcal{F}''_{-1}$

while twisting on K sends $\mathcal{F}''$ to $\mathcal{F}''_{+1}$

so we can realize any framing

Proof of Lemma 5:

Check back later!